

# Some bidouble planes with $p_g = q = 0$ and $4 \leq K^2 \leq 7$

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## Abstract

We construct smooth minimal bidouble planes  $S$  of general type with  $p_g = 0$  and  $K^2 = 4, \dots, 7$  having involutions  $i_1, i_2, i_3$  such that  $S/i_1$  is birational to an Enriques surface,  $S/i_3$  is rational and the bicanonical map of  $S$  is not composed with  $i_1, i_2$  and is composed with  $i_3$ .

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## 1 Introduction

Smooth minimal surfaces of general type with  $p_g = q = 0$  have been studied by several authors in the last years, but a classification is still missing. We refer the surveys [MP2] and [BCP] for information on these surfaces.

There is, to my knowledge, only one example of a smooth minimal surface  $S$  of general type with  $p_g = 0$  and  $K^2 = 7$  ([In]). This surface has an alternative description as a bidouble cover of a rational surface ([MP1]). Its bicanonical map is of degree two onto a rational surface and is not composed with the other two involutions  $i_1, i_2$  associated with the covering. Moreover,  $S/i_1$  and  $S/i_2$  are also rational (cf. [LS]).

In this paper we consider the case where  $S$  has an involution  $i$  such that the bicanonical map is not composed with  $i$  and  $S/i$  is not a rational surface. We construct examples with  $K^2 = 4, \dots, 7$  such that  $S/i$  is birational to an Enriques surface. This answers a question of Lee and Shin ([LS]) about the existence of the cases with  $K^2 = 5, 6, 7$  and  $S/i$  birational to an Enriques surface. In all cases  $S$  has another involution  $j$  such that  $S/j$  is rational and the bicanonical map of  $S$  is composed with  $j$ .

The paper is organized as follows. First we recall some facts on involutions. Secondly we note that minor modifications to [Ri1, Theorems 7, 8 and 9] give a list of possibilities for the branch curve in the quotient surface  $S/i$ . Then we construct some examples as double covers of an Enriques surface obtained as a quotient of a Kummer surface. In Section 4.2 we describe these surfaces as bidouble covers of the plane. Finally we give some other bidouble plane examples.

## Notation

We work over the complex numbers; all varieties are assumed to be projective algebraic. An *involution* of a surface  $S$  is an automorphism of  $S$  of order 2. We say that a map is *composed with an involution*  $i$  of  $S$  if it factors through the double cover  $S \rightarrow S/i$ . A  $(-2)$ -curve or *nodal curve*  $N$  on a surface is a curve

isomorphic to  $\mathbb{P}^1$  such that  $N^2 = -2$ . An  $(m_1, m_2, \dots)$ -point of a curve, or point of type  $(m_1, m_2, \dots)$ , is a singular point of multiplicity  $m_1$ , which resolves to a point of multiplicity  $m_2$  after one blow-up, etc. The rest of the notation is standard in Algebraic Geometry.

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## 2 General facts on involutions

The following is according to [CM].

Let  $S$  be a smooth minimal surface of general type with an involution  $i$ . Since  $S$  is minimal of general type, this involution is biregular. The fixed locus of  $i$  is the union of a smooth curve  $R''$  (possibly empty) and of  $t \geq 0$  isolated points  $P_1, \dots, P_t$ . Let  $S/i$  be the quotient of  $S$  by  $i$  and  $p : S \rightarrow S/i$  be the projection onto the quotient. The surface  $S/i$  has nodes at the points  $Q_i := p(P_i)$ ,  $i = 1, \dots, t$ , and is smooth elsewhere. If  $R'' \neq \emptyset$ , the image via  $p$  of  $R''$  is a smooth curve  $B''$  not containing the singular points  $Q_i$ ,  $i = 1, \dots, t$ . Let now  $h : V \rightarrow S$  be the blow-up of  $S$  at  $P_1, \dots, P_t$  and set  $R' = h^*(R'')$ . The involution  $i$  induces a biregular involution  $\tilde{i}$  on  $V$  whose fixed locus is  $R := R' + \sum_1^t h^{-1}(P_i)$ . The quotient  $W := V/\tilde{i}$  is smooth and one has a commutative diagram:

$$\begin{array}{ccc} V & \xrightarrow{h} & S \\ \pi \downarrow & & \downarrow p \\ W & \xrightarrow{g} & S/i \end{array}$$

where  $\pi : V \rightarrow W$  is the projection onto the quotient and  $g : W \rightarrow S/i$  is the minimal desingularization map. Notice that

$$A_i := g^{-1}(Q_i), \quad i = 1, \dots, t,$$

are  $(-2)$ -curves and  $\pi^*(A_i) = 2 \cdot h^{-1}(P_i)$ .

Set  $B' := g^*(B'')$ . Since  $\pi$  is a double cover with branch locus  $B' + \sum_1^t A_i$ , it is determined by a line bundle  $L$  on  $W$  such that

$$2L \equiv B := B' + \sum_1^t A_i.$$

**Proposition 1** ([CM], [CCM]) *The bicanonical map of  $S$  (given by  $|2K_S|$ ) is composed with  $i$  if and only if  $h^0(W, \mathcal{O}_W(2K_W + L)) = 0$ .*

### 3 List of possibilities

Let  $P$  be a minimal model of the resolution  $W$  of  $S/i$ , let  $\rho : W \rightarrow P$  be the corresponding projection and denote by  $\overline{B}$  the projection  $\rho(B)$ .

**Theorem 2** (cf. [Ri1]) *Let  $S$  be a smooth minimal surface of general type with  $p_g = 0$  having an involution  $i$  such that the bicanonical map of  $S$  is not composed with  $i$  and  $S/i$  is not rational.*

*With the previous notation, one of the following holds:*

a)  $P$  is an Enriques surface and:

·  $\overline{B}^2 = 0$ ,  $t - 2 = K_S^2 \in \{2, \dots, 7\}$ ,  $\overline{B}$  has a  $(3, 3)$ -point or a 4-uple point and at most one double point.

b)  $\text{Kod}(P) = 1$  and:

·  $K_P \overline{B} = 2$ ,  $\overline{B}^2 = -12$ ,  $t - 2 = K_S^2 \in \{2, \dots, 8\}$ ,  $\overline{B}$  has at most two double points, or

·  $K_P \overline{B} = 4$ ,  $\overline{B}^2 = -16$ ,  $t = K_S^2 \in \{4, \dots, 8\}$ ,  $\overline{B}$  is smooth.

c)  $\text{Kod}(P) = 2$  and:

·  $K_S^2 = 2K_P^2$ ,  $K_P^2 = 1, \dots, 4$ ,  $\overline{B}$  is a disjoint union of four  $(-2)$ -curves, or

·  $K_P \overline{B} = 2$ ,  $K_P^2 = 1$ ,  $\overline{B}^2 = -12$ ,  $t = K_S^2 \in \{4, \dots, 8\}$ ,  $\overline{B}$  has at most one double point, or

·  $K_P \overline{B} = 2$ ,  $K_P^2 = 2$ ,  $\overline{B}^2 = -12$ ,  $t + 2 = K_S^2 \in \{6, 7, 8\}$ ,  $\overline{B}$  is smooth.

Moreover there are examples for a), b) and c).

**Proof :** This follows from the proof of [Ri1, Theorems 7, 8 and 9] taking in account that:

- $p_g(P) = q(P) = 0$  (because  $p_g(P) \leq p_g(S)$ ,  $q(P) \leq q(S)$ );
- $h^0(W, \mathcal{O}_W(2K_W + L)) \leq \frac{1}{2}K_W^2 + 2$  (see [Ri1, Proposition 4, b)]);
- $K_S^2 \neq 9$  (see [DMP, Theorem 4.3]);
- We can have  $\overline{B}^2 > 0$  (unlike the case  $p_g = q = 1$ ).

Examples for a) and b) are given below. Rebecca Barlow ([Ba]) has constructed a surface of general type with  $p_g = 0$  and  $K^2 = 1$  containing an even set of four disjoint  $(-2)$ -curves. This gives an example for c).

### 4 Examples

#### 4.1 $S/i$ birational to an Enriques surface

Consider the involution of  $\mathbb{P}^1 \times \mathbb{P}^1$

$$j : [x : y, a : b] \mapsto [y : x, b : a]$$

and denote by  $f, g$  the projections onto the first and second factors, respectively. Let  $F_1, \dots, F_4, G_1, \dots, G_4$  be fibres of  $f, g$  such that

$$C := F_1 + \dots + F_4 + G_1 + \dots + G_4$$

is preserved by  $j$  and does not contain the fixed points  $[1 : \pm 1, 1 : \pm 1]$  of  $j$ .

Let

$$\pi : Q \rightarrow \mathbb{P}^1 \times \mathbb{P}^1$$

be the double cover with branch locus  $C$  and let  $k$  be the corresponding involution. It is well known that  $Q$  is a Kummer surface and

$$E := Q/k \circ j$$

is an Enriques surface with 8 nodes.

#### 4.1.1 $\overline{B}$ with a 4-uple point

Let  $D \subset \mathbb{P}^1 \times \mathbb{P}^1$  be a generic curve of bi-degree  $(1, 2)$  tangent to  $C$  at smooth points  $p_1, p_2$  of  $C$  such that  $p_2 = j(p_1)$ . The pullback  $\pi^*(D + j(D)) \subset Q$  is a reduced curve with two 4-uple points, corresponding to the  $(2, 2)$ -points of  $D + j(D)$  (which are tangent to the branch curve  $C$ ). These points are identified by the involution  $k \circ j$ , thus the projection of  $\pi^*(D + j(D))$  into  $E$  is a reduced curve  $\overline{B'}$  with one 4-uple point.

Now let  $\tilde{E}$  be the minimal smooth resolution of the Enriques surface  $E$ ,  $A_1, \dots, A_8 \subset \tilde{E}$  be the nodal curves corresponding to the nodes of  $E$  and  $\overline{B'} \subset \tilde{E}$  be the strict transform of  $\overline{B'}$ . If  $D$  does not contain one of the 16 double points of  $C$ , the divisor

$$\overline{B} := \overline{B'} + \sum_{i=1}^8 A_i$$

is reduced, divisible by 2 in the Picard group and satisfies  $\overline{B}^2 = 0$ . Let  $S$  be the smooth minimal model of the double cover of  $\tilde{E}$  ramified over  $\overline{B}$ . One can show that  $S$  is a surface of general type with  $p_g = 0$  and  $K_S^2 = 6$ . Moreover,  $D$  can be chosen through one or two double points of  $C$ . This provides examples with  $K_S^2 = 5$  or 4, corresponding to branch curves

$$\overline{B'} + \sum_{i=1}^7 A_i \quad \text{or} \quad \overline{B'} + \sum_{i=1}^6 A_i.$$

#### 4.1.2 $\overline{B}$ with a $(3, 3)$ -point

Let  $D_1$  be a curve of bi-degree  $(0, 1)$  through  $p$ ,  $D_2$  be a general curve of bi-degree  $(1, 1)$  through  $p$  and  $j(p)$  and set  $D := D_1 + D_2$ . Then  $D + j(D)$  is a reduced curve with triple points at  $p$  and  $j(p)$ . Now we proceed as in Section 4.1.1. In this case the branch curve  $\overline{B} \subset \tilde{E}$  has a  $(3, 3)$ -point instead of a 4-uple point. This gives an example of a surface of general type  $S$  with  $p_g = 0$  and  $K^2 = 7$  (notice that the resolution of the  $(3, 3)$ -point gives rise to an additional nodal curve in the branch locus). As above,  $D$  can be chosen containing one or two double points of  $C$ , providing examples with  $K_S^2 = 6$  or 5.

## 4.2 Bidouble plane description

Here we obtain the examples of Section 4.1 as bidouble covers of the plane.

### 4.2.1 Construction

Let  $T_1, \dots, T_4 \subset \mathbb{P}^2$  be distinct lines through a point  $p$  and  $C_1, C_2$  be distinct smooth conics tangent to  $T_1, T_2$  at points  $p_1, p_2 \neq p$ , respectively. The smooth minimal model  $\tilde{E}$  of the double cover of  $\mathbb{P}^2$  with branch locus  $T_1 + \dots + T_4 + C_1 + C_2$  is an Enriques surface with 8 disjoint nodal curves  $A_1, \dots, A_8$ , which correspond to the 8 double points of

$$G := T_3 + T_4 + C_1 + C_2.$$

Now let  $p_3$  be a generic point in  $T_3$  and consider the pencil  $l$  generated by  $2H_i + T_i$ ,  $i = 1, 2, 3$ , where  $H_i$  is a conic through  $p_i$  tangent to  $T_j, T_k$  at  $p_j, p_k$ , for each permutation  $(i, j, k)$  of  $(1, 2, 3)$ . Let  $L$  be a generic element of  $l$ . Notice that the quintic curve  $L$  contains  $p$ , it has a  $(2, 2)$ -point at  $p_i$  and the intersection number of  $L$  and  $T_i$  at  $p_i$  is 4,  $i = 1, 2, 3$ .

The strict transform of  $L$  in  $\tilde{E}$  is a reduced curve  $\overline{B}'$  with a 4-uple point (at the pullback of  $p_3$ ) such that the divisor

$$\overline{B} := \overline{B}' + \sum_{i=1}^8 A_i$$

is reduced, satisfies  $\overline{B}^2 = 0$  and is divisible by 2 in the Picard group (because  $L + T_1$  is divisible by 2). Let  $S$  be the smooth minimal model of the double cover of  $\tilde{E}$  ramified over  $\overline{B}$ . One can verify that  $K_S^2 = 6$ . As in Section 4.1.1, choosing  $L$  through 1 or 2 double points of  $T_3 + T_4 + C_1 + C_2$  one obtains examples with  $K_S^2 = 5$  or 4, respectively.

To obtain a branch curve  $\overline{B} \subset \tilde{E}$  with a  $(3, 3)$ -point as in Section 4.2, it suffices to change the  $(2, 2)$ -point of the quintic  $L$  at  $p_3$  to an ordinary triple point. In this case  $L$  is the union of a conic through  $p_3$  with a cubic having a double point at  $p_3$ . Choosing  $C_1$  and  $C_2$  so that  $L$  passes through 0, 1 or 2 double points of  $T_3 + T_4 + C_1 + C_2$  one obtains examples with  $K_S^2 = 7, 6$  or 5.

### 4.2.2 Involutions on $S$

We refer [Ca] or [Pa] for information on bidouble covers.

Each surface  $S$  constructed in Section 4.2.1 is the smooth minimal model of the bidouble cover of  $\mathbb{P}^2$  determined by the divisors

$$\begin{aligned} D_1 &:= L, \\ D_2 &:= T_1 + C_1 + C_2, \\ D_3 &:= T_2 + T_3 + T_4. \end{aligned}$$

Let  $i_g$  be the involution of  $S$  corresponding to  $D_j + D_k$ , for each permutation  $(g, j, k)$  of  $(1, 2, 3)$ . We have that  $S/i_1$  is birational to an Enriques surface,  $S/i_3$  is a rational surface and the bicanonical map of  $S$  is not composed with  $i_1, i_2$  and is composed with  $i_3$ . Moreover,  $S$  has an hyperelliptic fibration of genus 3.

We omit the proof for these facts: it is similar to the one given in [Ri3] for an example with  $K_S^2 = 3$ .

### 4.3 More bidouble planes

In the examples above,  $S/i_1$  is birational to an Enriques surface with 8 disjoint  $(-2)$ -curves, corresponding to the 8 nodes of the sextic  $G = T_3 + T_4 + C_1 + C_2$ , which contains 2 lines. Now we give examples with  $G$  containing only one line and with  $G$  without lines.

#### 4.3.1 $G$ with one line, $K_S^2 = 4, 5, 6$

Let  $T_1, T_2, T_3$  and  $L$  be as in Section 4.2.1 and  $p_4$  be a smooth point of  $L$ . There exists a plane curve  $J$  of degree 5 through  $p$  with  $(2, 2)$ -points tangent to  $T_1, T_2, L$  at  $p_1, p_2, p_4$ , respectively (notice that we are imposing 19 conditions to a linear system of dimension 20; such a curve can be easily computed using the Magma function `LinSys` given in [Ri2]).

Let  $S$  be the smooth minimal model of the bidouble cover of  $\mathbb{P}^2$  determined by the divisors

$$\begin{aligned} D_1 &:= L, \\ D_2 &:= T_3, \\ D_3 &:= T_1 + T_2 + J. \end{aligned}$$

Notice that the double plane with branch locus  $D_2 + D_3$  is an Enriques surface  $E$  with 6 disjoint nodal curves  $A_1, \dots, A_6$  (two of them are contained in the pullback of  $p_4$ ) and that the strict transform  $\widehat{L}$  of  $L$  in  $E$  has a 4-uple point at the pullback of  $p_3$ . Moreover, the divisor  $\overline{B} := \widehat{L} + \sum_1^6 A_i$  satisfies  $\overline{B}^2 = 0$  and is even (because  $L + T_3$  is even). This gives an example for Theorem 2, a) with  $K_S^2 = 4$ .

To obtain an example with  $K_S^2 = 5$  it suffices to choose the quintic  $J$  with a triple point at  $p_4$  instead of a  $(2, 2)$ -point. In this case  $J$  is the union of a conic with a singular cubic. Here the Enriques surface contains 7 disjoint nodal curves, three of them contained in the pullback of  $p_4$ .

Finally, choosing  $L$  with a triple point at  $p_3$  one obtains  $\widehat{L} \subset E$  with a  $(3, 3)$ -point. This gives examples for Theorem 2, a) with  $K_S^2 = 5, 6$ .

#### 4.3.2 $G$ without lines, $K_S^2 = 4$

Consider, in affine plane, the points  $p_0, \dots, p_5$  with coordinates  $(0, 0)$ ,  $(1, 1)$ ,  $(-1, 1)$ ,  $(2, 3)$ ,  $(-2, 3)$ ,  $(0, 5)$ , respectively, and let  $T_{ij}$  be the line through  $p_i, p_j$ . Let  $C_1$  be the conic tangent to  $T_{01}, T_{02}$  at  $p_1, p_2$  which contains  $p_5$  and let  $C_2$  be the conic tangent to  $T_{01}, T_{02}$  at  $p_1, p_2$  which contains  $p_3, p_4$ . Let  $l$  be the linear system generated by  $T_{01} + T_{02} + 2T_{34}$  and  $T_{03} + T_{04} + C_2$ .

The element  $Q$  of  $l$  through  $p_5$  is an irreducible quartic curve with double points at  $p_0, p_3, p_4$  and tangent to  $T_{01}, T_{02}$  at  $p_1, p_2$ . Moreover, because of the symmetry with respect to  $T_{05}$ , the line  $H$  tangent to  $Q$  at  $p_5$  is horizontal.

There is a cubic  $F$  through  $p_0, p_3, p_4$  tangent to  $T_{01}, T_{02}, H$  at  $p_1, p_2, p_5$ , respectively (notice that we are imposing 9 conditions to a linear system of dimension 9). One can verify that  $F$  contains no line, thus it is irreducible.

The surface  $S$  is the smooth minimal model of the bidouble cover of  $\mathbb{P}^2$  determined by the divisors

$$\begin{aligned} D_1 &:= C_1 + F, \\ D_2 &:= T_{01}, \\ D_3 &:= T_{02} + C_2 + Q. \end{aligned}$$

Notice that the double plane with branch locus  $D_2 + D_3$  is an Enriques surface  $E$  with 6 disjoint nodal curves (contained in the pullback of the triple points  $p_3, p_4$  of  $D_3$ ) and that the strict transform of  $D_1$  in  $E$  has a 4-uple point (at the pullback of  $p_5$ ). The double plane with branch locus  $D_1 + D_3$  is a surface with Kodaira dimension 1. This gives an example for Theorem 2, a), b) with  $K_S^2 = 4$ .

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